BBSI: “Lab work” for 5/22/07: Linear Algebra

1) **Some basic properties of matrix eigenvalues/eigenvectors**. Consider the matrix

$$A_0 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

a) Find the eigenvalues of this matrix analytically.

b) Use Mathcad to compute the eigenvalues and eigenvectors of this matrix numerically.

c) Now consider the matrix:

$$A = \begin{pmatrix} 2 & -1 & .2 \\ -1 & 2 & -1 \\ .2 & -1 & 2 \end{pmatrix}$$

Use Mathcad to compute the eigenvalues and eigenvectors of this matrix numerically. Verify that all eigenvectors are mutually orthogonal. Also verify that the eigenvectors returned by Mathcad are unit normalized (have length 1).

d) If the eigenvalues of A are denoted $\lambda_1, \lambda_2, \lambda_3$, let $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$ [i.e., the diagonal matrix of eigenvalues]. Furthermore, let T be the 3x3 matrix of unit normalized eigenvectors (one eigenvector in each column). Construct the matrix A as: $A = T\Lambda T^t$.

e) Show that any power of the matrix A can be calculated as $A^n = T\Lambda^n T^t$, where $\Lambda^n = diag(\lambda_1{}^n, \lambda_2{}^n, \lambda_3{}^n)$. Use this relation to calculate A^3. [Hint: you can check your result vs. a direct call to Mathcad for this quantity.]

f) Calculate exp(A); calculate exp(-A). [Check that exp(A)exp(-A)=1, where 1 is the 3x3 identity matrix.]

2) **Linear least squares fit: operational formulae.** Given a data set consisting of N (x,y) pairs, it is often useful to fit this data to a specified functional form. Let us consider polynomial fitting functions: $f(x) = \sum_{k=0}^{p} c_k x^k$, where p is the order of the polynomial. The problem is to determine the optimal coefficients $c_0, c_1, ..., c_p$.

The standard way to proceed is to minimize the sum of the squares of the deviation (error) in the fit to each data point. That is, consider

$$E = \sum_{j=1}^{N} [f(x_j) - y_j]^2$$

where (x_j, y_j) is the j'th data point, of which there are N total. Then the "best" choice of coefficients in the fitting function is the set which minimizes the value of E>0. This criterion is called a "least squares fit".

It can be shown that if *f(x)* is the p'th order polynomial indicated above, then the optimal coefficients are determined via the expression $\vec{c} = M^{-1}\vec{b}$, where $\vec{c}$ is the (p+1) dimensional vector containing the desired coefficients, i.e. $\vec{c} = (c_0, c_1, ..., c_p)^t$, M is a (p+1)x(p+1) dimensional matrix with the elements $M_{jk} = \sum_{i=1}^{N} x_i^{j+k}$ [indexing $j,k = 0,1,...,p$] , and $\vec{b}$ is the (p+1) dimensional vector with elements $b_j = \sum_{i=1}^{N} x_i^j y_i$ [indexing similarly].

Consider the following data set with N=10: $x_i = 0.2i$, i=1,2,…,10, and

y_i = 1.76, 1.60, 1.48, 1.31, 1.23, 1.06, 1.01, 0.892, 0.807, 0.765

Use Mathcad to determine the least squares fit quadratic polynomial (p=2) that best approximates this data. Note: Fig. 1 plots the data set and the associated least-squares quadratic fit curve. Your task is to find the values of c_0, c_1, c_2 which determine the dashed line!

3) **Linear least squares fit: derivation.** Derive the formula for the optimal coefficients $\vec{c} = (c_0, c_1, ..., c_p)^t$ stated in Problem 2. Hint: Given the assumed functional form of $f(x)$ noted in Problem 2, then the error for given values of the coefficients $\vec{c}$ is given by:

$$E(\vec{c}) = \sum_{j=1}^{N} [\sum_{k=0}^{p} c_k x_j^k - y_j]^2$$

The value of E is minimized when $\partial E / \partial c_n = 0$ for $n = 0,1,\ldots, p$.

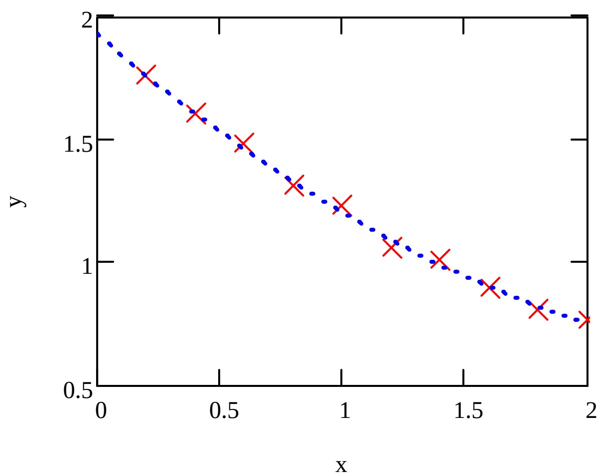

Fig. 1: (x_j, y_j) data points in Problem 2 (x's), and 2nd order polynomial least square fit to them (dashed line).