①

Review of Linear Algebra [for BBSI; 5/03]

Basic operations:

Matrix/vector multiplication:

$$\begin{bmatrix}1&2&3\\1&0&1\\5&7&2\end{bmatrix}\begin{bmatrix}-1\\3\\0\end{bmatrix}=\begin{bmatrix}5\\-1\\16\end{bmatrix}$$

matrix/matrix multiplication:

$$N_1\left\{\overset{\underset{\sim}{M_1}}{\begin{bmatrix}1&2&1\\-2&0&3\end{bmatrix}}\right.\overset{\underset{\sim}{M_2}}{\begin{bmatrix}4&0\\2&-1\\-1&-2\end{bmatrix}}\Big\}N_2 = N_1\left\{\overset{\underset{\sim}{M_2}}{\begin{bmatrix}7&-4\\-11&-6\end{bmatrix}}\right.$$

↑

must have $M_1 = N_2$

dot product of a vector: Let $\vec{v}=\begin{pmatrix}-1\\3\\0\end{pmatrix}$; then $\vec{u}\cdot\vec{v} = (1\,2\,1)\begin{pmatrix}-1\\3\\0\end{pmatrix} = 5 = (-1\;3\;0)\begin{pmatrix}1\\2\\1\end{pmatrix} = \vec{v}\cdot\vec{u}$

$\vec{u}=\begin{pmatrix}1\\2\\1\end{pmatrix}$

$\updownarrow$

Note:

$\vec{u}\cdot\vec{u} = |\vec{u}|^2 = 6 = l_u^2$; l = length of vector $u \geq 0$

Linear Systems of Eq.:

given parameter

1 Eq. in 1 unknown: (linear) $\quad y = mx + b \Rightarrow \frac{1}{m}(y-b) = x$ ← "Solve for x"

2 linear Eq. in 2 unknowns:

$$\begin{aligned} 2x_1 + x_2 &= 4 \\ 2[\,-x_1 + 2x_2 &= 3\,] \\ \hline 5x_2 = 10 \Rightarrow x_2 &= 2 \Rightarrow x_1 = 1 \end{aligned}$$

Try: 3 linear Eq. in 3 unknowns:

$$\begin{pmatrix}1&2&4\\-1&0&1\\3&-2&-4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}8\\0\\-16\end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 & 8 \end{pmatrix} \times 3$$
$$\begin{matrix} 0 & 2 & 5 & 8 \\ 3 & -2 & -4 & -16 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 4 & 8 \\ 0 & 2 & 5 & 8 \end{matrix}$$
$$\begin{bmatrix} 0 & 8 & 16 & 40 \end{bmatrix} \times \frac{1}{4}$$

$$\underbrace{\begin{matrix} 1 & 2 & 4 & 8 \\ 0 & 2 & 5 & 8 \\ 0 & 0 & 1 & -2 \end{matrix}}_{\text{Gauss-Jordan Elimination}} \Rightarrow \begin{matrix} x_1 = \\ x_2 = \\ x_3 = \end{matrix} \begin{pmatrix} -2 \\ 9 \\ -2 \end{pmatrix}$$

More generally:

$$\begin{pmatrix} 1 & 2 & 4 \\ -1 & 0 & 1 \\ 3 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \qquad (*)$$

Indeed:

$$\begin{pmatrix} 1 & 2 & 4 \\ -1 & 0 & 1 \\ 3 & -2 & -4 \end{pmatrix} \underbrace{\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}}_{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \frac{1}{8} \begin{pmatrix} 2 & 0 & 2 \\ -1 & -16 & -5 \\ 2 & 8 & 2 \end{pmatrix} = \boxed{A^{-1}} \quad ; \text{ since } \quad A A^{-1} = \mathbb{1} = A^{-1} A$$

see (*); check column by column

concept works for any NxN square matrix(+)

(+) though A^{-1} may not exist(!)

③

Now note: $\vec{A} \cdot \vec{x} = \vec{y} \leftarrow \underline{\text{any}}\ \vec{y} \Rightarrow \boxed{\vec{x} = A^{-1} \cdot \vec{y}}$

for example, in our case $$A^{-1}\begin{pmatrix} 8 \\ 0 \\ -8 \end{pmatrix} = \cancel{\tfrac{1}{8}}\begin{pmatrix} 2 & 0 & 2 \\ -1 & -16 & -5 \\ 2 & 8 & 2 \end{pmatrix}\begin{pmatrix} \cancel{8}1 \\ 0 \\ \underset{-2}{\cancel{-16}} \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \\ -2 \end{pmatrix} \quad \checkmark$$

//

~~N×N matrix~~

Determinant: det(A) = scalar

Next, define, for 1x1 matrix $A = a$; $\det A = a$

Now, go to 2x2 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$|A| \equiv \det A = a_{11} \cdot (-1)^{1+1} \det(a_{22}) + a_{12}(-1)^{1+2} \det(a_{21}) = a_{11}a_{22} - a_{12}a_{21} \leftarrow \text{"go across 1st row"}$$

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & a_{22} \end{bmatrix} \qquad \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}} \end{bmatrix}$$

$$= a_{12}(-1)^{1+2}\det(a_{21}) + a_{22}(-1)^{2+2}\det(a_{11}) = a_{22}a_{11} - a_{12}a_{21} \leftarrow \text{"go down 2nd column"}$$

Move to 3x3 case:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} = 2(-1)^{1+2}\underbrace{\overset{\equiv A_{12}}{\begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix}}}_{1} + (-2)(-1)^{\boxed{3+2}}\underbrace{\overset{\equiv A_{32}}{\begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix}}}_{5} = \boxed{8}$$

(*)

(*)

$A_{32} \leftarrow$ "minor" of a_{32} ; "cofactor" of a_{32} is $(-1)^{3+2}A_{32}$

" go down 2nd column "

$$= \sum_{i=1}^{3} a_{i2}(-1)^{i+2} A_{i2}$$

could sum down any row or any column; get same result(!)

Now, a determinant-based formula for the **inverse of a square matrix**:

Consider:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 1 \\ 3 & -2 & -4 \end{bmatrix}$$

1st calculate the cofactor matrix:

cofactor of A_{13}

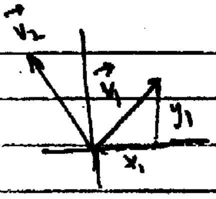
$$\text{cof } A = \begin{bmatrix} (+)\begin{vmatrix} 0 & 1 \\ -2 & -4 \end{vmatrix} & (-)\begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} & (+)\begin{vmatrix} -1 & 0 \\ 3 & -2 \end{vmatrix} \\ (-)\begin{vmatrix} 2 & 4 \\ -2 & -4 \end{vmatrix} & (+)\begin{vmatrix} 1 & 4 \\ 3 & -4 \end{vmatrix} & (-)\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \\ (+)\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} & (-)\begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix} & (+)\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 2 \\ 0 & -16 & 8 \\ 2 & -5 & 2 \end{bmatrix}$$

Then:

$$\boxed{A^{-1} = \frac{1}{\det A} (\text{cof } A)^t} = \boxed{\frac{1}{8}\begin{bmatrix} 2 & 0 & 2 \\ -1 & -16 & -5 \\ 2 & 8 & 2 \end{bmatrix}}$$

Linear transformation:



$$T\,(V_1) = V_2$$

↑
Transformation

for any scale constant
↓

Properties of a linear transformation : i) $T(\alpha V_1) = \alpha\, T(V_1)$

ii) $T(V_1 + V_2) = T(V_1) + T(V_2)$

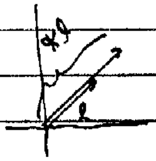
NB: Any linear transformation can be represented by a matrix :

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ - & - & - \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Note : T can take a vector from 3D and map it into 2D, etc.

Linear transformations with geometrical significance:

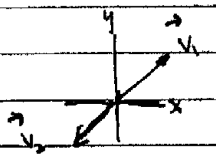
i) length scaling :



$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

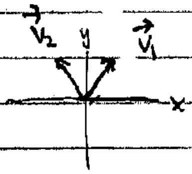
$\downarrow\ \alpha = -1$

ii) Inversion :



$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

iii) Reflection about (say) y axis :



$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

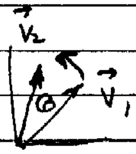
Note: this matrix representation of the "reflection about y axis" linear transformation $\underset{\sim}{T}$ can be obtained by determining what T does to each basis vector $\hat{x}, \hat{y}$:

$$T(\hat{x}) = -\hat{x} \quad \text{or} \quad \begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}-1\\0\end{pmatrix} ; \quad T(\hat{y}) = \hat{y} \quad \text{or} \quad \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

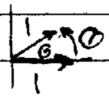
Thus

$$T \leftrightarrow \left[\begin{pmatrix}-1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \right]$$

iv) rotation by Θ:

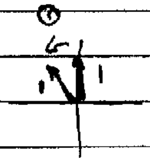


Consider what happens to $\begin{pmatrix}1\\0\end{pmatrix}$:



$$\begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{T} \begin{pmatrix}\cos\Theta\\ \sin\Theta\end{pmatrix}$$

what happens to $\begin{pmatrix}0\\1\end{pmatrix}$:



$$\begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{T} \begin{pmatrix}-\sin\Theta\\ \cos\Theta\end{pmatrix}$$

Thus

$$\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{bmatrix}\cos\Theta & -\sin\Theta\\ \sin\Theta & \cos\Theta\end{bmatrix} \begin{pmatrix}x_1\\x_2\end{pmatrix}$$

Check that this T is "length preserving"

$$l^2 = \vec{V}_1 \cdot \vec{V}_1 = (x_1 \, y_1) \begin{pmatrix}x_1\\y_1\end{pmatrix} \; ; \text{ what is } \vec{V}_2 \cdot \vec{V}_2 = (x_2 \, y_2) \begin{pmatrix}x_2\\y_2\end{pmatrix} = (x_1 y_1) \underbrace{\overset{\underset{\sim}{T}^t}{\begin{pmatrix}\cos\Theta & \sin\Theta\\ -\sin\Theta & \cos\Theta\end{pmatrix}} \overset{\underset{\sim}{T}}{\begin{pmatrix}\cos\Theta & -\sin\Theta\\ \sin\Theta & \cos\Theta\end{pmatrix}}}_{\begin{pmatrix}1&0\\0&1\end{pmatrix}} \begin{pmatrix}x_1\\y_1\end{pmatrix}$$

↑ l = vector length

$$\checkmark \; = l^2$$

← check it out!

Note: $\underset{\approx}{T}^t = \underset{\approx}{T}^{-1}$ ← such matrices are called "orthogonal"

[corresponding to an "orthogonal transformation"]

↖ length preserving

Eigenvalues + Eigenvectors $\underset{\sim}{A}$

Consider the (abstract) problem $\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \lambda\begin{pmatrix}x\\ y\end{pmatrix}$ [1]

Find $\lambda, \begin{pmatrix}x\\ y\end{pmatrix} \leftarrow \begin{pmatrix}x\\ y\end{pmatrix} \neq \begin{pmatrix}0\\ 0\end{pmatrix}$ (*)

Rewrite [1]: $\begin{pmatrix}2-\lambda & -1\\ -1 & 2-\lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0$ [1']

for (*) to be satisfied, must have $\underbrace{\det\begin{pmatrix}2-\lambda & -1\\ -1 & 2-\lambda\end{pmatrix}}_{\text{secular determinant}} = 0$

thus:

$$(2-\lambda)^2 - 1 = 0 \Rightarrow (\lambda-2) = \pm 1 \Rightarrow \boxed{\lambda_+ = 3 \; ; \; \lambda_- = 1}$$

Determine the corresponding eigenvectors:

Substitute back into [1']:

for $\underline{\lambda_- = 1}$: $\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0$; let $x = 1 \Rightarrow y = 1$; thus $\boxed{\begin{pmatrix}x_-\\ y_-\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}}$

[N.B.: $\overset{\text{any value}}{\alpha}\begin{pmatrix}1\\ 1\end{pmatrix}$ is also an eigenvector corresponding to λ_- [the eigenvector is only determined up to a normalization constant]

and

for $\underline{\lambda_+ = 3}$ $\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0$; let $x = 1 \Rightarrow y = -1$

thus: $\boxed{\begin{pmatrix}x_+\\ y_+\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}}$

NB: i) Upon letting "$\underline{x} = 1$" (← for example), we are left w/ N-1 linear eqs. in N-1 variables ← solve by matrix inversion! ← for an N×N matrix $\underset{\sim}{A}$

ii) It may happen that $x = 0$ (in reality); then procedure i) will give "nonsense"; if so, set "$\underline{y} = 1$" and try again!

Reconstruction of $\underset{\approx}{A}$ in terms of its eigenvalues and eigenvectors:

Let:

$$N\{\underbrace{[A]}_{N}[v] = \lambda[v]\}N$$

these do not all have to be different

There will by N eigenvector $\vec{V}_1, \vec{V}_2, \dots, \vec{V}_N$ corresponding to $\lambda_1, \lambda_2, \dots, \lambda_N$

All of this info can be summarized as (illustrating with the N=3 case)

$$3\{\underbrace{[A]}_{3}\underbrace{\left[(\vec{V}_1)(\vec{V}_2)(\vec{V}_3)\right]}_{3}\} = \underbrace{\left[(\vec{V}_1)(\vec{V}_2)(\vec{V}_3)\right]}_{\underset{\approx}{T}}\underbrace{\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}}_{\underset{\approx}{\Lambda}} = \left[\lambda_1(\vec{V}_1)\;\; \lambda_2(\vec{V}_2)\;\; \lambda_3(\vec{V}_3)\right] \quad \checkmark$$

Thus:

$$\underbrace{\underset{\approx}{T}^{-1}\underset{\approx}{A}\,\underset{\approx}{T} = \underset{\approx}{\Lambda}}_{\underset{\approx}{A}\text{ is "diagonilized" by }\underset{\approx}{T}}\;;\text{ or: }\quad \underbrace{\underset{\approx}{A} = \underset{\approx}{T}\,\underset{\approx}{\Lambda}\,\underset{\approx}{T}^{-1}}_{A\text{ is reconstructed from its eigenvalues/eigenvectors}} \qquad (2]$$

NB: there is one simplifying feature of the above analysis when (real-valued) $\underset{\approx}{A}$ is symmetric $[A_{ij} = A_{ji}]$.

It can be shown that the eigenvectors of a real symmetric matrix are mutually orthogonal. That is (illustrating for N=3):

$$\vec{V}_1 \cdot \vec{V}_2 = \vec{V}_1 \cdot \vec{V}_3 = \vec{V}_2 \cdot \vec{V}_3 = 0$$

Thus, by unit normalizing the eigenvectors, i.e. $\vec{V}_1 \cdot \vec{V}_1 = \vec{V}_2 \cdot \vec{V}_2 = \vec{V}_3 \cdot \vec{V}_3 = 1$

... it follows that $\underset{\approx}{T}^{-1} = \underset{\approx}{T}^{t}$ [← $\underset{\approx}{T}$ defined above]

Proof:

$$\underbrace{\begin{bmatrix}(\;\vec{V}_1\;)\\(\;\vec{V}_2\;)\\(\;\vec{V}_3\;)\end{bmatrix}}_{\underset{\approx}{T}^{t}}\underbrace{\left[(\vec{V}_1)(\vec{V}_2)(\vec{V}_3)\right]}_{\underset{\approx}{T}} = \begin{pmatrix}\overset{1}{\vec{V}_1\cdot\vec{V}_1} & \overset{0}{\vec{V}_1\cdot\vec{V}_2} & \overset{0}{\vec{V}_1\cdot\vec{V}_3}\\ \vec{V}_2\cdot\vec{V}_1 & \vec{V}_2\cdot\vec{V}_2 & \vec{V}_2\cdot\vec{V}_3\\ \vec{V}_3\cdot\vec{V}_1 & \vec{V}_3\cdot\vec{V}_2 & \vec{V}_3\cdot\vec{V}_3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \checkmark$$

Thus, for real-symmetric matrices $\underline{\underline{A}}$, Eq. [2] becomes:

$$\underline{\underline{T}}^t \, \underline{\underline{A}} \, \underline{\underline{T}} = \underline{\underline{\Lambda}} \; ; \; \underline{\underline{A}} = \underline{\underline{T}} \, \underline{\underline{\Lambda}} \, \underline{\underline{T}}^t \qquad [2']$$

no need to compute inverse of $\underline{\underline{T}}$ (!)

Example: Apply [2'] to the 2x2 case worked out above:

$$\underline{\underline{T}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \; ; \; \underline{\underline{\Lambda}} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

unit normalize the eigenvectors!

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \overset{?}{=} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\underline{\underline{T}}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}_{\underline{\underline{\Lambda}}} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}}_{\underline{\underline{T}}^t} \qquad (\leftrightarrow \text{ here } \underline{\underline{T}}^t = \underline{\underline{T}}) \qquad [3]$$

yes (check it out)

not generally the case (!)

<u>Geometric</u> <u>Relevance</u>: Diagonalization of a Quadratic Form

Consider

$$V(x,y) = (x\,y) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2(x^2 - xy + y^2)$$

quadratic form simplifies (no "cross terms")

$$= (xy) \, \underline{\underline{T}} \, \underline{\underline{\Lambda}} \, \underbrace{\underline{\underline{T}}^t \begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} u \\ v \end{pmatrix}} = \underbrace{(u\,v) \, \underline{\underline{\Lambda}} \begin{pmatrix} u \\ v \end{pmatrix}}_{\text{no } uv \text{ cross term}} = u^2 + 3v^2$$

use [2'], [3]

for specific details relevant to $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

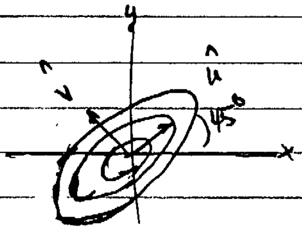
Flushing out details...

$$\begin{pmatrix}u\\v\end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}}_{\underset{=}{T}^t}\begin{pmatrix}x\\y\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}x\\y\end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}}_{\underset{\approx}{T}}\begin{pmatrix}u\\v\end{pmatrix}$$

Thus $\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

Geometrically :



Reference: Matrices with Applications, H. G. Campbell (Meredith Corp., NY, 1968)