

## BBSI: “Lab work” for 5/24/06: Linear Algebra

1) Consider the matrix

$$A_0 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

- Find the eigenvalues of this matrix analytically.
- Use Mathcad to compute the eigenvalues and eigenvectors of this matrix numerically.
- Now consider the matrix:

$$A = \begin{pmatrix} 2 & -1 & .2 \\ -1 & 2 & -1 \\ .2 & -1 & 2 \end{pmatrix}$$

Use Mathcad to compute the eigenvalues and eigenvectors of this matrix numerically. Verify that all eigenvectors are mutually orthogonal. Also verify that the eigenvectors returned by Mathcad are unit normalized (have length 1).

- If the eigenvalues of  $A$  are denoted  $\lambda_1, \lambda_2, \lambda_3$ , let  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$  [i.e., the diagonal matrix of eigenvalues]. Furthermore, let  $T$  be the 3x3 matrix of unit normalized eigenvectors (one eigenvector in each column). Construct the matrix  $A$  as:  $A = T\Lambda T^t$ .
- Show that any power of the matrix  $A$  can be calculated as  $A^n = T\Lambda^n T^t$ , where  $\Lambda^n = \text{diag}(\lambda_1^n, \lambda_2^n, \lambda_3^n)$ . Use this relation to calculate  $A^3$ . [Hint: you can check your result vs. a direct call to Mathcad for this quantity.]
- Calculate  $\exp(A)$ ; calculate  $\exp(-A)$ . [Check that  $\exp(A)\exp(-A)=1$ , where 1 is the 3x3 identity matrix.]

2) Given a data set consisting of  $N$   $(x,y)$  pairs, it is often useful to fit this data to a specified functional form. Let us consider polynomial fitting functions:  $f(x) = \sum_{k=0}^p c_k x^k$ , where  $p$  is the order of the polynomial. The problem is to determine the optimal coefficients  $c_0, c_1, \dots, c_p$ .

The standard way to proceed is to minimize the sum of the squares of the deviation (error) in the fit to each data point. That is, consider

$$E = \sum_{j=1}^N [f(x_j) - y_j]^2$$

where  $(x_j, y_j)$  is the  $j$ 'th data point, of which there are  $N$  total. Then the "best" choice of coefficients in the fitting function is the set which minimizes the value of  $E > 0$ . This criterion is called a "least squares fit".

a) Show that if  $f(x)$  is the  $p$ 'th order polynomial indicated above, then the optimal coefficients are determined via the expression  $\vec{c} = M^{-1}\vec{b}$ , where  $\vec{c}$  is the  $(p+1)$  dimensional vector containing the desired coefficients, i.e.  $\vec{c} = (c_0, c_1, \dots, c_p)^t$ ,  $M$  is a

$(p+1) \times (p+1)$  dimensional matrix with the elements  $M_{jk} = \sum_{i=1}^N x_i^{j+k}$  [indexing

$j, k = 0, 1, \dots, p$ ], and  $\vec{b}$  is the  $(p+1)$  dimensional vector with elements  $b_j = \sum_{i=1}^N x_i^j y_i$

[indexing similarly].

b) Consider the following data set with  $N=10$ :  $x_i = 0.2i$ ,  $i=1, 2, \dots, 10$ , and

$y_i = 1.76, 1.60, 1.48, 1.31, 1.23, 1.06, 1.01, 0.892, 0.807, 0.765$

Use the analysis in a) plus Mathcad (to do the arithmetic) to determine the least squares fit quadratic polynomial ( $p=2$ ) that best approximates this data. Plot the data and the associated least-squares quadratic fit curve on the same graph.